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Automorphically-invariant ideals satisfying multilinear identities, and group-theoretic applications

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Abstract

Let A be an arbitrary (not necessarily associative or commutative) algebra over a field K. It is proved that if A has an ideal of finite codimension r satisfying a multilinear identity $f \equiv 0$, then A also has an ideal satisfying the same identity $f \equiv 0$ that is invariant under all automorphisms of A and has finite codimension bounded in terms of r and f. The result is stronger in characteristic zero, where f need not be multilinear.

As a corollary, it is proved that if a locally nilpotent torsion-free group G has a normal subgroup H satisfying a multilinear commutator identity $\varkappa(H) \equiv 1$ with quotient G/H of finite rank r, then G also has a characteristic subgroup C satisfying the same identity $\varkappa(C) \equiv 1$ with quotient G/C of finite rank bounded in terms of r and $\varkappa$.

An example shows that the main result cannot be extended to algebras not over fields, even to Lie algebras over integers. An analogous example shows that the result on characteristic nilpotent subgroups with quotients of finite rank, which was proved by the authors earlier in torsion-free and periodic cases, cannot be extended to mixed nilpotent groups.

Keywords: Non-associative algebra; Multilinear identity; Nilpotent; Lie ring; Ideal; Automorphism; Locally nilpotent torsion-free group; Rank

1. Introduction

In [4] we proved that if a Lie algebra L over a field has a nilpotent ideal of nilpotency class c and of finite codimension r, then L also has an automorphically-invariant (that is, invariant under

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all automorphisms of L) nilpotent ideal of class $\leqslant c$ and of finite codimension bounded in terms of r and c. In the present paper we extend this result to arbitrary (not necessarily associative, or commutative, or finite-dimensional) algebras over a field and arbitrary multilinear identities.

Theorem 1. *If an algebra A over a field has an ideal of finite codimension r satisfying a multilinear identity $f \equiv 0$, then A also has an automorphically-invariant ideal that satisfies the same identity $f \equiv 0$ and has finite codimension bounded above in terms of r and f.*

Multilinear identities define many important varieties of algebras, for example, soluble and nilpotent varieties. Since in characteristic zero every identity is equivalent to several multilinear identities, we have the following stronger result in this case.

Theorem 2. *If an algebra A over a field of characteristic zero has an ideal of finite codimension r satisfying an arbitrary identity $f \equiv 0$, then A also has an automorphically-invariant ideal that satisfies the same identity $f \equiv 0$ and has finite codimension bounded above in terms of r and f.*

It is worth mentioning that an automorphically-invariant ideal of a locally nilpotent Lie algebra of characteristic zero is also invariant under all of its derivations. Thus Theorem 2 can be applied in a situation, where a locally nilpotent Lie algebra M of characteristic zero has a "subideal" I—an ideal of an ideal L of M: if I satisfies an identity $f \equiv 0$ and has finite codimension r in L, then there also exists an ideal K of M of finite (f, r)-bounded codimension in L satisfying the same identity $f \equiv 0$. (Henceforth, say, "(a, b)-bounded" abbreviates "bounded in terms of a and b.")

Instead of a single identity in Theorems 1, 2 one can consider a variety given by several identities: then an automorphically-invariant ideal in the same variety can be obtained as the corresponding intersection and its codimension will be bounded in terms of r and all these identities. The functions bounding the codimension in Theorems 1, 2 can be estimated from above explicitly, although we do not write down this estimate.

We produce an example showing that these results cannot be extended to algebras not over fields—even to Lie algebras over integers. In fact, our example is a nilpotent Lie ring that has an abelian ideal with quotient of rank 2 as an additive group but does not have an automorphically-invariant abelian ideal with quotient of finite rank. Recall that a group has rank $\leqslant r$ if every finitely generated subgroup can be generated by r elements.

Virtually "the same" example provides a negative answer to a question on groups. Earlier we proved in [4] that if a group G contains a normal nilpotent subgroup N of class c that is either torsion-free or periodic and has quotient of finite rank r, then G has a characteristic (that is, automorphically-invariant) subgroup C with quotient of (r, c)-bounded rank that is nilpotent of class at most c. It was natural to try to drop the assumption that N is either torsion-free or periodic. We now show that this is actually impossible even for N abelian. (A similar example was independently and almost simultaneously constructed by H. Smith.)

The specialization of Theorem 1 to Lie algebras implies a similar result for locally nilpotent torsion-free groups, via the Mal'cev correspondence given by the Baker–Campbell–Hausdorff formula. For groups, the role of codimension is taken by "co-rank"—the rank of the quotient by a normal subgroup. Earlier the special case of this result for nilpotent subgroups was proved in [4].

Theorem 3. *If a locally nilpotent torsion-free group G has a normal subgroup H satisfying a multilinear commutator identity $\varkappa(H)=1$ such that G/H has finite rank r, then G also has a characteristic subgroup C satisfying the same identity $\varkappa(C)=1$ such that G/C has finite rank bounded above in terms of r and $\varkappa$.*

It remains unclear whether the same kind of results can be proved for arbitrary torsion-free or periodic groups with a subgroup of finite "co-rank" satisfying an arbitrary multilinear commutator identity (additional conditions of being torsion-free or periodic, at least for the subgroup, are necessary as shown by the aforementioned example in this paper). On the other hand, we have previously proved in [3] a general result for subgroups of bounded index: if a group G has a subgroup H of finite index m satisfying a multilinear commutator identity $\varkappa(H)=1$, then G has also a characteristic subgroup C of finite $(m,\varkappa)$-bounded index satisfying the same identity $\varkappa(C)=1$. Both for groups and algebras over a field (even Lie algebras) it is also unclear whether the same kind of results can be proved for non-multilinear identities—of course, apart from algebras in characteristic zero covered by Theorem 2.

Our results in [3,4], on "transforming" a normal nilpotent subgroup into a characteristic subgroup that has the same nilpotency class and bounded index or co-rank, proved to be very useful in certain induction arguments; in particular, they enabled us to finish in [2,3,6] the solutions of certain problems on groups with almost regular automorphisms. We can now use Theorem 3 to improve one of our results in [5], where H was merely a normal subgroup with the same properties, by replacing "normal" by "characteristic."

Corollary 1. *Suppose that a locally nilpotent torsion-free group G admits an automorphism φ of finite order n such that the fixed-point subgroup $C_G(\varphi)$ has finite rank r. Then G has a soluble characteristic subgroup H of derived length bounded above in terms of n such that the quotient G/H has finite rank bounded above in terms of r and n.*

Of course, due to Theorem 1 we also have a similar strengthening of our generalization of Kreknin's theorem in [5], by way of inserting the additional condition of the ideal in the conclusion being automorphically-invariant.

Corollary 2. *If a Lie algebra L admits an automorphism φ of finite order n with fixed-point subalgebra of finite dimension $\dim C_L(\varphi)=m$, then L has a soluble automorphically-invariant ideal of derived length bounded above in terms of n and of finite codimension bounded above in terms of m and n.*

2. Preliminaries

An arbitrary (not necessarily associative, or commutative, or finite-dimensional) algebra A over a field K is a vector space over K with multiplication satisfying the axioms

$$(a+b)c=ac+bc, \qquad a(b+c)=ab+ac, \qquad \alpha(ab)=(\alpha a)b=a(\alpha b),$$

where $a,b,c,\in A$ and $\alpha\in K$.

We denote by $K\langle X\rangle$ a free algebra on a set of free generators $X = \{x_1, x_2, \dots\}$. Elements of $K\langle X\rangle$ can be regarded as polynomials in non-associative non-commuting variables x_i. An element $f(x_1, \dots, x_n) \in K\langle X\rangle$ is called *multilinear* if it is linear in all variables, that is,

$$f(x_1, \dots, \alpha y_j + \beta z_j, \dots, x_n) = \alpha f(x_1, \dots, y_j, \dots, x_n) + \beta f(x_1, \dots, z_j, \dots, x_n)$$

for any $\alpha, \beta \in K$, $x_i, y_j, z_j \in X$, $1 \leqslant j \leqslant n$. As usual, an algebra A *satisfies an identity* $f \equiv 0$ for $f = f(x_1, x_2, \dots, x_n) \in K\langle X\rangle$ if $f(a_1, a_2, \dots, a_n) = 0$ for any $a_1, a_2, \dots, a_n \in A$, where, of course, $f(a_1, a_2, \dots, a_n)$ is the value of f at the a_i (the image of $f(x_1, x_2, \dots, x_n)$ under the homomorphism extending the mapping $x_i \longrightarrow a_i$, $i = 1, \dots, n$).

An *ideal* B of an algebra A is a subspace such that $ab \in B$ and $ba \in B$ for any $a \in A$ and $b \in B$. We say for short that an ideal B of A is *automorphically-invariant* if B is invariant under all automorphisms of A.

Let $X = \{x_1, x_2, \dots\}$ be a set of non-associative non-commuting variables. *Non-associative words in the variables x_i* are defined by induction on weight. The words of weight 1 are the x_i themselves. The words of weight $w > 1$ are the expressions (uv), where u, v are words of weight w_1, w_2 with $w = w_1 + w_2$. The *multiweight* of a word u is the tuple $(n_1, n_2, \dots)$ where n_i is the number of occurrences of x_i in u. Clearly, the words in X form a basis of the free algebra $K\langle X\rangle$ on free generators X. Elements of the form αu, where $\alpha \in K$ and u is a word in the x_i, are called *monomials*. It is easy to see that a polynomial $f(x_1, \dots, x_n) \in K\langle X\rangle$ is multilinear if and only if it is multihomogeneous of multiweight $(1, \dots, 1)$, that is, it is a linear combination of words of weight n each involving exactly one occurrence of x_i for $i = 1, \dots, n$.

It is well known that over a field of characteristic zero any identity $f \equiv 0$ is equivalent to a system of multilinear identities; it is also important for us that the weights of these identities and their number are bounded in terms of f.

Proposition 1. (See, for example, [8, Ch. 1].) *Let A be an algebra over a field K of characteristic zero. Let $f \in K\langle X\rangle$ be a non-associative polynomial and let $f = \sum_i^l u_i$ be the decomposition of f into a sum of homogeneous components u_i. Then A satisfies the identity $f \equiv 0$ if and only if A satisfies all the identities $u_i \equiv 0$. Moreover, A satisfies $u_i \equiv 0$ if and only if A satisfies the linearization of u_i, which is a multilinear polynomial of the same degree as u_i.*

In this section we also prove a technical proposition in linear algebra that is used later in the proof of Theorem 1. Although the statement may seem rather involved, it is precisely what we shall need later.

Definition. Let A be a vector space over a field, and $N_1, \dots, N_m$ subspaces of A. Suppose that $A = M \oplus R$, where M and R are subspaces of A, and let π denote the projection of A onto M with respect to this direct sum. In the tensor power

$$\underbrace{M \otimes \cdots \otimes M}_{t}$$

we define the *block-diagonal projection span of $N_1, \dots, N_m$ with respect to the projection π* to be

$$\underbrace{\pi(N_1) \otimes \cdots \otimes \pi(N_1)}_{t} + \cdots + \underbrace{\pi(N_m) \otimes \cdots \otimes \pi(N_m)}_{t}. \tag{1}$$

The following lemma will be used in situations where the dimension n of M is relatively small as compared with the number m of subspaces N_i. Then this lemma will enable us to reduce the number of summands in (1) to some (n,t)-bounded value. The proof is based on a simple dimension argument.

Lemma 1. *Let A be a vector space with subspaces $N_1, \dots, N_m$. Suppose that*

$$A = M \oplus R, \tag{2}$$

where $\dim M = n$, and let π be the projection onto M with respect to (2). Let t be a positive integer. Then one can choose a subset $\mathcal{P}$ of the index set $\{1, \dots, m\}$ with number of elements $|\mathcal{P}|$ at most n^t such that the block-diagonal projection span of $N_1, \dots, N_m$ in

$$\underbrace{M \otimes \cdots \otimes M}_{t}$$

with respect to π is spanned by the subspaces $\underbrace{\pi(N_j) \otimes \cdots \otimes \pi(N_j)}_{t}$ with $j \in \mathcal{P}$, that is,

$$\sum_{j=1}^{m} \underbrace{\pi(N_j) \otimes \cdots \otimes \pi(N_j)}_{t} = \sum_{j \in \mathcal{P}} \underbrace{\pi(N_j) \otimes \cdots \otimes \pi(N_j)}_{t}. \tag{3}$$

Proof. Since $\dim M = n$, the tensor power

$$\underbrace{M \otimes \cdots \otimes M}_{t}$$

has dimension n^t. Therefore the block-diagonal span

$$\underbrace{\pi(N_1) \otimes \cdots \otimes \pi(N_1)}_{t} + \cdots + \underbrace{\pi(N_m) \otimes \cdots \otimes \pi(N_m)}_{t} \tag{4}$$

also has dimension $\leqslant n^t$. Adding consecutively those of the subspaces

$$\underbrace{\pi(N_i) \otimes \cdots \otimes \pi(N_i)}_{t}$$

that strictly increase the span we shall need at most n^t steps to exhaust the whole space (4); the corresponding indices i form the required subset $\mathcal{P}$ such that (3) holds. $\square$

A repeated "augmented" application of the above lemma gives the requisite technical proposition.

Proposition 2. *Let $c \geqslant 2$ be an integer and let A be a vector space with subspaces $N_1, \dots, N_m$ each of codimension $\leqslant r$ in A. Then there exists an increasing chain of subsets of indices*

$$\varnothing = \mathcal{P}_0 \subseteq \mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \cdots \subseteq \mathcal{P}_c \subseteq \{1, \dots, m\}$$

such that for each $t = 1, \dots, c$ the subset $\mathcal{P}_t$ has (r, c)-bounded number of elements and

$$\sum_{j=1}^{m} \underbrace{\pi_t(N_j) \otimes \cdots \otimes \pi_t(N_j)}_{t} = \sum_{j \in \mathcal{P}_t} \underbrace{\pi_t(N_j) \otimes \cdots \otimes \pi_t(N_j)}_{t}, \tag{5}$$

where π_t is the projection onto some complement M_t of $\bigcap_{i \in \mathcal{P}_{t-1}} N_i$ in A with respect to

$$A = M_t \oplus \bigcap_{i \in \mathcal{P}_{t-1}} N_i. \tag{6}$$

That is, the block-diagonal span of $N_1, \dots, N_m$ in

$$\underbrace{M_t \otimes \cdots \otimes M_t}_{t}$$

with respect to the projection π_t onto M_t defined by (5) *is spanned by the subspaces $\underbrace{\pi_t(N_j) \otimes \cdots \otimes \pi_t(N_j)}_{t}$ with $j \in \mathcal{P}_t$.*

Proof. First we define by induction (r, c)-bounded numbers q_t for $t = 1, \dots, c$:

$$q_1 = r + 1, \qquad q_t = q_{t-1} + (rq_{t-1})^t.$$

With advance knowledge we can reveal that for each t the required subset $\mathcal{P}_t$ will have at most q_t elements.

We construct the $\mathcal{P}_t$ by induction on t.

Case $t = 1$. Since $\dim(A/N_1) \leqslant r$, the sum $N_1 + \cdots + N_m$ is equal to the sum of at most $r + 1$ of the N_i including N_1, so we define $\mathcal{P}_1$ to be the set of the indices of the N_i involved:

$$N_1 + \cdots + N_m = \sum_{i \in \mathcal{P}_1} N_i, \tag{7}$$

where the subset $\mathcal{P}_1$ has at most $q_1 = r + 1$ elements. Since $\mathcal{P}_0 = \varnothing$, here $M_1 = A$ and $\pi_1(N_j) = N_j$, so (7) is precisely the required equality (6) for $t = 1$.

Induction step $t > 1$. Suppose that the required subsets of indices $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \cdots \subseteq \mathcal{P}_{t-1}$ are already constructed with $|\mathcal{P}_i| \leqslant q_i$. We choose an arbitrary complement M_t of $\bigcap_{i \in \mathcal{P}_{t-1}} N_i$ in A:

$$A = M_t \oplus \bigcap_{i \in \mathcal{P}_{t-1}} N_i.$$

Since $\dim(A/N_i) \leqslant r$ for all i and $\mathcal{P}_{t-1}$ has at most q_{t-1} elements, the quotient $A/\bigcap_{i \in \mathcal{P}_{t-1}} N_i$ has dimension $\leqslant rq_{t-1}$, so that $\dim M_t \leqslant rq_{t-1}$. By Lemma 1 applied to $M = M_t$ and $R = \bigcap_{i \in \mathcal{P}_{t-1}} N_i$ with $n = \dim M \leqslant rq_{t-1}$, there is a subset $\mathcal{P} \subseteq \{1, \dots, m\}$ such that $|\mathcal{P}| \leqslant n^t$ and

$$\sum_{j=1}^{m} \underbrace{\pi_t(N_j) \otimes \cdots \otimes \pi_t(N_j)}_{t} = \sum_{j \in \mathcal{P}} \underbrace{\pi_t(N_j) \otimes \cdots \otimes \pi_t(N_j)}_{t}.$$

This equality will of course remain valid if we replace $\mathcal{P}$ by $\mathcal{P}_t = \mathcal{P} \cup \mathcal{P}_{t-1}$, simply in order to have the required inclusion $\mathcal{P}_{t-1} \subseteq \mathcal{P}_t$. Then $|\mathcal{P}_t| \leqslant q_t$ and the required equality (6) holds. $\square$

3. Proof of the main result

In this section we prove Theorem 1. Recall that we are given an algebra A (over a field K) which has an ideal I of finite codimension r satisfying the identity $f \equiv 0$, where $f = f(x_1, x_2, \ldots, x_c) \in K\langle X\rangle$ is a non-associative multilinear polynomial in c variables with coefficients in K. We must find an automorphically-invariant ideal J of A of finite (r,c)-bounded codimension satisfying the identity $f \equiv 0$. The idea of the proof is as follows. The intersection

$$\bigcap_{\alpha \in \operatorname{Aut} A} I^{\alpha}$$

of all images of I under automorphisms of A is an automorphically-invariant ideal of A. If this intersection is equal to the intersection of a small, (r,c)-bounded, number of ideals of the form I^{α}, then this intersection has (r,c)-bounded codimension and is the required automorphically-invariant ideal. In the opposite case we argue by contradiction.

Definition. We say that a collection of sets $\{N_1, \ldots, N_m\}$ has *essential intersection* if

$$\bigcap_{i=1}^{m} N_i \neq \bigcap_{i \neq j} N_i \quad \text{for each } j.$$

It is clear that for a collection of sets with essential intersection every subcollection of it also has essential intersection—we shall often use this property without any special references.

In the proof of Theorem 1 we can assume from the outset that I is maximal by inclusion among the ideals of codimension $\leqslant r$ satisfying the identity $f \equiv 0$. We shall prove that then there cannot be too many ideals I^{α} with essential intersection. Thus, Theorem 1 will be proved if we prove the following proposition, which can be applied with the N_i being some ideals I^{α} in a set with essential intersection.

Proposition 3. *Let A be an algebra with ideals $N_1, \ldots, N_m$, and let $f(x_1, \ldots, x_c) \in K\langle X\rangle$ be a multilinear polynomial of degree c. Suppose that*

(a) *each N_i satisfies the identity $f \equiv 0$ and has codimension $\leqslant r$ in A;*
(b) *the set $N_1, \ldots, N_m$ has essential intersection.*

If $m \geqslant s(r,c)$ for a certain (r,c)-bounded number $s(r,c)$, then there exists $k \in \{1, \ldots, m\}$ such that the ideal $N_k + \bigcap_{i \neq k} N_i$ satisfies the identity $f \equiv 0$.

Of course, a more rigorous formulation would be to claim the existence of the function $s(r,c)$, which will in fact be defined recursively in the proof (and we shall see that it can be estimated from above explicitly). Note that the ideal $N_k + \bigcap_{i \neq k} N_i$ is strictly larger than N_k because the collection of the N_i has essential intersection; this is what provides the required contradiction in the proof of Theorem 1.

For any subsets $B_1, \dots, B_c \subseteq A$ we denote by $f(B_1, \dots, B_c)$ the subspace spanned by the set $\{f(b_1, \dots, b_c) \mid b_1 \in B_1, \dots, b_c \in B_c\}$.

Proof of Proposition 3. To strengthen the induction hypothesis we actually prove the following assertion.

Lemma 2. *Let $\mathcal{P}_1, \dots, \mathcal{P}_c$ be the subsets constructed in Proposition 2. Then for each $t = 1, \dots, c$ and each $k = 1, \dots, m$*

$$f\left(\dots, N_k, \dots, \bigcap_{i \in \mathcal{P}(t)} N_i, \dots, N_k, \dots, \bigcap_{i \in \mathcal{P}(t)} N_i, \dots, N_k, \dots\right) = 0, \tag{8}$$

where among the arguments of the polynomial f there are t entries N_k and $c - t$ entries $\bigcap_{i \in \mathcal{P}(t)} N_i$ in any order.

Equality (8) implies that for any $k \notin \mathcal{P}(t)$ the subspace

$$f\left(\dots, N_k, \dots, \bigcap_{i \neq k} N_i, \dots, N_k, \dots, \bigcap_{i \neq k} N_i, \dots, N_k, \dots\right)$$

is also trivial, since $\bigcap_{i \neq k} N_i \leqslant \bigcap_{i \in \mathcal{P}(t)} N_i$ for $k \notin \mathcal{P}(t)$. When this assertion is proved for all $t = 1, 2, \dots, c$, we shall obtain that the ideal $N_k + \bigcap_{i \neq k} N_i$ satisfies the identity $f \equiv 0$ for every $k \notin \mathcal{P}(c)$. Indeed, since f is multilinear,

$$f\underbrace{\left(N_k + \bigcap_{i \neq k} N_i, \dots, N_k + \bigcap_{i \neq k} N_i\right)}_{c}$$

is the sum of the subspaces (8) plus the subspace

$$f\underbrace{\left(\bigcap_{i \neq k} N_i, \dots, \bigcap_{i \neq k} N_i\right)}_{c},$$

which is clearly trivial because every N_i satisfies the identity $f \equiv 0$. Thus Proposition 3 will be proved.

Proof of Lemma 2. Induction on t.

Case $t = 1$. Here, N_k occurs once in (8), while the other $c - 1$ entries are $\bigcap_{i \in \mathcal{P}(1)} N_i$. By Proposition 2,

$$\sum_{i=1}^{m} N_i = \sum_{j \in \mathcal{P}(1)} N_j.$$

Then for every k

$$f\Bigl(\bigcap_{i\in\mathcal{P}(1)} N_i,\dots N_k,\dots,\bigcap_{i\in\mathcal{P}(1)} N_i\Bigr)\subseteq\sum_{j\in\mathcal{P}(1)} f\Bigl(\bigcap_{i\in\mathcal{P}(1)} N_i,\dots,N_j,\dots,\bigcap_{i\in\mathcal{P}(1)} N_i\Bigr)$$
$$\subseteq\sum_{j\in\mathcal{P}(1)} f(N_j,\dots,N_j,\dots,N_j)=0, \tag{9}$$

since each ideal N_j satisfies $f\equiv 0$. (Note that this reasoning is independent of the position of N_k among the arguments of f.)

Case $t = 2$. We consider for clarity one more initial step of induction before describing formally the induction step. At this second step we are interested in subspaces of the form

$$f\Bigl(\dots,N_k,\dots,\bigcap_{i\in\mathcal{P}_2} N_i,\dots,\bigcap_{i\in\mathcal{P}_2} N_i,\dots,,N_k,\dots,\bigcap_{i\in\mathcal{P}_2} N_i,\dots\Bigr) \tag{10}$$

with exactly two occurrences of N_k and the other $c-2$ entries being $\bigcap_{i\in\mathcal{P}_2} N_i$. It is convenient to fix the positions of N_k among the arguments of f in (10), but the reasoning is independent of the choice of these positions. The crucial idea is to consider both entries of N_k in (10) modulo $\bigcap_{i\in\mathcal{P}_1} N_i$. We now elaborate on this remark in certain linear terms and use Proposition 2.

For any $u_1,u_2\in A$ and $v_1,\dots,v_{c-2}\in\bigcap_{i\in\mathcal{P}_1} N_i$ we have the multilinear map

$$\varphi_{u_1,u_2}:(v_1,\dots,v_{c-2})\longmapsto f(v_1,\dots,u_1,\dots,u_2,\dots,v_{c-2})$$

(where u_1,u_2 are in the positions of N_k in (10)), which naturally defines a linear map denoted by the same symbol

$$\varphi_{u_1,u_2}:\underbrace{\bigcap_{i\in\mathcal{P}_1} N_i\otimes\cdots\otimes\bigcap_{i\in\mathcal{P}_1} N_i}_{c-2}\longrightarrow f(\underbrace{A,\dots,A}_{c}).$$

The map $(u_1,u_2)\longmapsto\varphi_{u_1,u_2}$ is also multilinear and therefore defines a linear map

$$\theta:A\otimes A\longrightarrow\mathrm{Hom}\Bigl(\underbrace{\bigcap_{i\in\mathcal{P}_1} N_i\otimes\cdots\otimes\bigcap_{i\in\mathcal{P}_1} N_i}_{c-2},\,f(\underbrace{A,\dots,A}_{c})\Bigr).$$

By Proposition 2 for some complement $M(2)$ of $\bigcap_{i\in\mathcal{P}_1} N_i$ in A,

$$A=M(2)\oplus\bigcap_{i\in\mathcal{P}_1} N_i, \tag{11}$$

and for the projection π_2 onto $M(2)$ with respect to (11), we have

$$\sum_{i=1}^{m}\pi_2(N_i)\otimes\pi_2(N_i)=\sum_{j\in\mathcal{P}_2}\pi_2(N_j)\otimes\pi_2(N_j). \tag{12}$$

Let $u_1, u_2 \in N_k$. To lighten the notation, we denote by $m_1 = \pi_2(u_1)$, $m_2 = \pi_2(u_2)$ the projections of u_1, u_2 onto $M(2)$, and by y_1, y_2 the projections of u_1, u_2 onto $\bigcap_{i \in \mathcal{P}_1} N_i$ with respect to (11). Then

$$\begin{aligned} u_1 \otimes u_2 &= (m_1 + y_1) \otimes (m_2 + y_2) \\ &= m_1 \otimes m_2 + m_1 \otimes y_2 + y_1 \otimes m_2 + y_1 \otimes y_2 \\ &= m_1 \otimes m_2 + u_1 \otimes y_2 + y_1 \otimes u_2 - y_1 \otimes y_2. \end{aligned}$$

By the linearity of θ we have

$$\begin{aligned} \theta(u_1 \otimes u_2) &= \theta(m_1 \otimes m_2) + \theta(u_1 \otimes y_2) + \theta(y_1 \otimes u_2) - \theta(y_1 \otimes y_2) \\ &= \varphi_{m_1,m_2} + \varphi_{u_1,y_2} + \varphi_{y_1,u_2} - \varphi_{y_1,y_2}. \end{aligned}$$

Since $y_1, y_2 \in \bigcap_{i \in \mathcal{P}_1} N_i$ and $u_1, u_2 \in N_k$, the linear maps φ_{u_1,y_2}, φ_{y_1,u_2}, and φ_{y_1,y_2} are trivial by (9). Therefore,

$$\theta(u_1 \otimes u_2) = \varphi_{m_1,m_2} = \theta(m_1 \otimes m_2). \tag{13}$$

By (12) the element $m_1 \otimes m_2 = \pi_2(u_1) \otimes \pi_2(u_2)$ is a sum of elements of the form $\pi_2(u_{i1}) \otimes \pi_2(u_{i2})$ where $u_{i1}, u_{i2} \in N_i$ for $i \in \mathcal{P}_2$:

$$m_1 \otimes m_2 = \sum_{i \in \mathcal{P}_2} \pi_2(u_{i1}) \otimes \pi_2(u_{i2}).$$

Then

$$\theta(m_1 \otimes m_2) = \sum_{i \in \mathcal{P}_2} \theta\bigl(\pi_2(u_{i1}) \otimes \pi_2(u_{i2})\bigr).$$

By (13) we have $\theta(\pi_2(u_{i1}) \otimes \pi_2(u_{i2})) = \theta(u_{i1} \otimes u_{i2})$ for all i. Hence

$$\varphi_{u_1,u_2} = \theta(u_1 \otimes u_2) = \theta(m_1 \otimes m_2) = \sum_{i \in \mathcal{P}_2} \theta(u_{i1} \otimes u_{i2}) = \sum_{i \in \mathcal{P}_2} \varphi_{u_{i1},u_{i2}},$$

where, recall, $u_{i1}, u_{i2} \in N_i$ for $i \in \mathcal{P}_2$.

Now let $v_1, \ldots, v_{c-2} \in \bigcap_{j \in \mathcal{P}_2} N_j$ and $u_1, u_2 \in N_k$. Since $\mathcal{P}_2 \supseteq \mathcal{P}_1$ and therefore

$$\bigcap_{i \in \mathcal{P}_2} N_i \leqslant \bigcap_{i \in \mathcal{P}_1} N_i,$$

by the above arguments we have

$$\begin{aligned} f(v_1, \dots, u_1, \dots, u_2, \dots, v_{c-2}) &= \varphi_{u_1,u_2}(v_1, \dots, v_{c-2}) \\ &= \sum_{i \in \mathcal{P}_2} \varphi_{u_{i1},u_{i2}}(v_1, \dots, v_{c-2}) \\ &= \sum_{i \in \mathcal{P}_2} f(v_1, \dots, u_{i1}, \dots, u_{i2}, \dots, v_{c-2}), \end{aligned}$$

where $u_{i1}, u_{i2} \in N_i$ for $i \in \mathcal{P}_2$. It remains to note that for each $i \in \mathcal{P}_2$

$$f(v_1, \dots, u_{i1}, \dots, u_{i2}, \dots, v_{c-2}) = 0,$$

since $v_1, \dots, v_{c-2} \in \bigcap_{j \in \mathcal{P}_2} N_j \subseteq N_i$ and $f(\underbrace{N_i, \dots, N_i}_{c}) = 0$. Thus,

$$f(v_1, \dots, u_1, \dots, u_2, \dots, v_{c-2}) = 0$$

for all $v_1, \dots, v_{c-2} \in \bigcap_{j \in \mathcal{P}_2} N_j$ and $u_1, u_2 \in N_k$.

Induction step $t > 2$. We now want to prove that

$$f\left(\dots, N_k, \dots, \bigcap_{i \in \mathcal{P}_t} N_i, \dots, N_k, \dots, \bigcap_{i \in \mathcal{P}_t} N_i, \dots, N_k, \dots\right) = 0, \tag{14}$$

where among the arguments of f there are t entries N_k and $c - t$ entries $\bigcap_{i \in \mathcal{P}_t} N_i$ in some order. Again, it is convenient to think of certain fixed positions of N_k among the arguments of f, but the reasoning is independent of the choice of these positions. As in the case of $t = 2$, the idea is to consider N_k modulo $\bigcap_{i \in \mathcal{P}_{t-1}} N_i$.

For any $u_1, \dots, u_t \in A$ and $v_1, \dots, v_{c-t} \in \bigcap_{i \in \mathcal{P}_{t-1}} N_i$ we define the multilinear map

$$\varphi_{u_1,\dots,u_t} : (v_1, \dots, v_{c-t}) \mapsto f(\dots, u_1, \dots, v_i, \dots, u_2, \dots, v_l, \dots),$$

where the entries u_i on the right are in the positions of the entries N_k in (14), and the v_j in the positions of $\bigcap_{i \in \mathcal{P}_t} N_i$. We have the corresponding linear map of the tensor product denoted by the same symbol

$$\varphi_{u_1,\dots,u_t} : \underbrace{\bigcap_{i \in \mathcal{P}_{t-1}} N_i \otimes \dots \otimes \bigcap_{i \in \mathcal{P}_{t-1}} N_i}_{c-t} \longrightarrow f(\underbrace{A, \dots, A}_{c}).$$

The map $(u_1, \dots, u_t) \mapsto \varphi_{u_1,\dots,u_t}$ is also multilinear and therefore defines a linear map

$$\theta : \underbrace{A \otimes \dots \otimes A}_{t} \longrightarrow \mathrm{Hom}\left(\underbrace{\bigcap_{i \in \mathcal{P}_{t-1}} N_i \otimes \dots \otimes \bigcap_{i \in \mathcal{P}_{t-1}} N_i}_{c-t}, f(\underbrace{A, \dots, A}_{c})\right).$$

By Proposition 2 for some complement M_t of $\bigcap_{i\in\mathcal{P}_{t-1}} N_i$ in A,

$$A = M_t \oplus \bigcap_{i\in\mathcal{P}_{t-1}} N_i, \tag{15}$$

and for the projection π_t onto M_t with respect to (15) we have

$$\sum_{i=1}^{m} \underbrace{\pi_t(N_i)\otimes\cdots\otimes\pi_t(N_i)}_{t} = \sum_{j\in\mathcal{P}_t} \underbrace{\pi_t(N_j)\otimes\cdots\otimes\pi_t(N_j)}_{t}. \tag{16}$$

Suppose that $u_1,\dots,u_t\in N_k$. By the induction hypothesis if at least one of the u_s belongs to $\bigcap_{i\in\mathcal{P}_{t-1}} N_i$, then $\varphi_{u_1,\dots,u_t}=0$.

To lighten the notation we denote by $m_i=\pi_t(u_i)$ the projections of the u_i onto M_t, and by y_i the projections of the u_i onto $\bigcap_{j\in\mathcal{P}_{t-1}} N_j$ with respect to (15). Using the expressions $u_i=m_i+y_i$ we represent the element $u_1\otimes\cdots\otimes u_t$ of the tensor product

$$\underbrace{A\otimes\cdots\otimes A}_{t}$$

as a sum of the element $m_1\otimes m_2\otimes\cdots\otimes m_t$ and a linear combination of elements of the form $w_1\otimes\cdots\otimes w_t$, where each w_i is either u_i or y_i and at least one of the w_i is y_i. By the induction hypothesis, $\theta(w_1\otimes\cdots\otimes w_t)=\varphi_{w_1,\dots,w_t}=0$ for all these summands $w_1\otimes\cdots\otimes w_t$, since $u_1,\dots,u_t\in N_k$ and $y_i\in\bigcap_{j\in\mathcal{P}_{t-1}} N_j$. By the linearity of θ we have

$$\theta(u_1\otimes\cdots\otimes u_t)=\theta(m_1\otimes\cdots\otimes m_t). \tag{17}$$

By (16) the element $m_1\otimes\cdots\otimes m_t=\pi_t(u_1)\otimes\cdots\otimes\pi_t(u_t)$ is a sum of elements of the form $\pi_t(u_{i1})\otimes\cdots\otimes\pi_t(u_{it})$, where $u_{i1},\dots,u_{it}\in N_i$ for $i\in\mathcal{P}_t$:

$$m_1\otimes\cdots\otimes m_t=\sum_{i\in\mathcal{P}_t}\pi_t(u_{i1})\otimes\cdots\otimes\pi_t(u_{it}).$$

Then

$$\theta(m_1\otimes\cdots\otimes m_t)=\sum_{i\in\mathcal{P}_t}\theta\bigl(\pi_t(u_{i1})\otimes\cdots\otimes\pi_t(u_{it})\bigr).$$

By (17) we have $\theta(\pi_t(u_{i1})\otimes\cdots\otimes\pi_t(u_{it}))=\theta(u_{i1}\otimes\cdots\otimes u_{it})$ for all i. Hence,

$$\begin{aligned}\varphi_{u_1,\dots,u_t}&=\theta(u_1\otimes\cdots\otimes u_t)=\theta(m_1\otimes\cdots\otimes m_t)\\&=\sum_{i\in\mathcal{P}_t}\theta(u_{i1}\otimes\cdots\otimes u_{it})\\&=\sum_{i\in\mathcal{P}_t}\varphi_{u_{i1},\dots,u_{it}},\end{aligned} \tag{18}$$

where, recall, $u_{i1},\dots,u_{it}\in N_i$ for $i\in\mathcal{P}_t$.

Now let $v_1,\dots,v_{c-t}\in\bigcap_{j\in\mathcal{P}_t} N_j$ and $u_1,\dots,u_t\in N_k$. By (18) we have

$$\begin{aligned} f(v_1, \dots, u_1, \dots, u_t, \dots, v_{c-t}) &= \varphi_{u_1,\dots,u_t}(v_1, \dots, v_{c-t}) \\ &= \sum_{i \in \mathcal{P}_t} \varphi_{u_{i1},\dots,u_{it}}(v_1, \dots, v_{c-t}) \\ &= \sum_{i \in \mathcal{P}_t} f(v_1, \dots, u_{i1}, \dots, u_{it}, \dots, v_{c-t}), \end{aligned}$$

where $u_{i1}, \dots, u_{it} \in N_i$ for $i \in \mathcal{P}_t$. Since $v_1, \dots, v_{c-t} \in \bigcap_{j \in \mathcal{P}_t} N_j \subseteq N_i$ for $i \in \mathcal{P}_t$ and $f(\underbrace{N_i, \dots, N_i}_{c}) = 0$, we have

$$f(v_1, \dots, u_{i1}, \dots, u_{it}, \dots, v_{c-t}) = 0.$$

Hence,

$$f(v_1, \dots, u_1, \dots, u_t, \dots, v_{c-t}) = 0$$

for all $v_1, \dots, v_{c-t} \in \bigcap_{j \in \mathcal{P}_t} N_j$ and $u_1, \dots, u_t \in N_k$, as required.

The proof of Lemma 2 is complete and therefore so are the proofs of Proposition 3 and Theorem 1. □

Theorem 2 follows immediately from Theorem 1 by Proposition 1. □

4. Locally nilpotent torsion-free groups

In this section we prove Theorem 3. The general idea is to embed the group into a radicable locally nilpotent torsion-free group, which is "category-equivalent" to a Lie algebra over $\mathbb{Q}$; then an application of Theorem 1 to this Lie algebra produces an appropriate ideal, which corresponds to the required subgroup. First we recall certain properties of locally nilpotent torsion-free groups and the aforementioned equivalence of the categories of locally nilpotent radicable torsion-free groups and locally nilpotent Lie algebras over $\mathbb{Q}$ (see [7] or [1]).

A group G is said to be radicable if it contains an nth root of any element $g \in G$ for any $n \in \mathbb{Z}$, that is, an element $h \in G$ such that $h^n = g$. Any locally nilpotent torsion-free group H can be embedded into a radicable locally nilpotent torsion-free group G by formally adjoining all roots of all non-trivial elements of H; then G is called the Mal'cev completion of H, which fact we denote as $G = \sqrt{H}$. Roots are unique in any locally nilpotent torsion-free group. The Mal'cev completion $\sqrt{H}$ is a unique (up to an isomorphism identical on H) radicable locally nilpotent torsion-free group all of whose elements are roots of elements of H. Every automorphism of H has a unique extension to $\sqrt{H}$. If N is a subgroup of H, then the Mal'cev completion of N is isomorphic to the set $\sqrt{N}$ of all roots of elements of N in H; if N is normal in H, then $\sqrt{N}$ is a normal subgroup of $\sqrt{H}$.

Let G be a radicable locally nilpotent torsion-free group. The same set G can be viewed as a locally nilpotent Lie algebra L over the field of rational numbers $\mathbb{Q}$ with the Lie ring operations given by the inversions of the Baker–Campbell–Hausdorff formula. In particular, the identity element of G is the zero element of L, and $a^r = ra$, where the left-hand side is the rth power, $r \in \mathbb{Q}$, of an element $a \in G$, while the right-hand side is the rth multiple of the same element $a \in L$. The group operations can be reconstructed by the Baker–Campbell–Hausdorff formula.

This is the Mal'cev correspondence—an equivalence of the categories of locally nilpotent radicable torsion-free groups and locally nilpotent Lie algebras over $\mathbb{Q}$. The automorphisms of G are automorphisms of L acting on the same set in the same way, and vice versa. Note that an automorphism of G as an abstract group is also an automorphism of G as a radicable group, since roots are unique in a torsion-free group.

From the explicit form of the Baker–Campbell–Hausdorff formula and its inversions it is easy to deduce that two elements $a, b \in G$ commute if and only if they commute in L. Indeed, the group commutator $[a,b]_G$ is a linear combination of Lie ring commutators in a and b (of weights $\geqslant 2$); and conversely, the Lie ring commutator $[a,b]_L$ is a product of rational powers of group commutators in a and b. (In both cases the sum and the product are finite, since the weights of non-trivial commutators in a, b do not exceed the nilpotency class of the subgroup/subalgebra generated by a and b.)

Definition. Let $x_1, x_2, \dots$ be group variables. *Multilinear commutators of weight* 1 *in the variables* x_i are the variables x_i themselves. By induction, *multilinear commutators of weight* $w > 1$ *in the variables* x_i are commutators of the form $\varkappa = [\varkappa_1, \varkappa_2]$, where $\varkappa_1$ and $\varkappa_2$ are multilinear commutators with disjoint sets of variables of weights w_1 and w_2 for $w = w_1 + w_2$.

A group G satisfies a multilinear commutator identity $\varkappa \equiv 1$ if and only if the commutator (verbal) subgroup $\varkappa(G)$, obtained by substituting the group G in place of all the variables in the commutator $\varkappa$, is trivial.

Multilinear commutator identities are inherited by Mal'cev completions, which is a consequence of the following lemma.

Lemma 3. *If H is an abstract subgroup of a radicable locally nilpotent torsion-free group G, and $\varkappa$ a multilinear commutator, then $\varkappa(\sqrt{H}) = \sqrt{\varkappa(H)}$.*

Proof. Induction on the weight of $\varkappa$. For weight 1 we have $\sqrt{\varkappa(H)} = \sqrt{H} = \varkappa(\sqrt{H})$. Now let $\varkappa = [\varkappa_1, \varkappa_2]$. By the induction hypothesis, $\sqrt{\varkappa_i(H)} = \varkappa_i(\sqrt{H})$. It is well known that in a torsion-free locally nilpotent group, roots of two elements commute if and only if the elements themselves commute. (One can easily prove this fact by using the Mal'cev correspondence: a^k and b^l commute in the group if and only if the same elements ka and lb commute in the Lie algebra, where obviously $[ka, lb]_L = kl[a,b]_L$.) Hence, $[\sqrt{\varkappa_1(H)}, \sqrt{\varkappa_2(H)}] \leqslant \sqrt{[\varkappa_1(H), \varkappa_2(H)]}$. But clearly $[\sqrt{\varkappa_1(H)}, \sqrt{\varkappa_2(H)}] \geqslant [\varkappa_1(H), \varkappa_2(H)]$, whence

$$\left[\sqrt{\varkappa_1(H)}, \sqrt{\varkappa_2(H)}\,\right] = \sqrt{\left[\sqrt{\varkappa_1(H)}, \sqrt{\varkappa_2(H)}\,\right]} \geqslant \sqrt{\left[\varkappa_1(H), \varkappa_2(H)\right]}.$$

Here we used the fact that in a locally nilpotent group the commutator subgroup $[A, B]$ of two radicable normal subgroups A, B is again radicable. Thus,

$$\varkappa(\sqrt{H}\,) = \left[\varkappa_1(\sqrt{H}\,), \varkappa_2(\sqrt{H})\,\right] = \left[\sqrt{\varkappa_1(H)}, \sqrt{\varkappa_2(H)}\,\right] = \sqrt{\left[\varkappa_1(H), \varkappa_2(H)\right]} = \sqrt{\varkappa(H)}. \qquad \square$$

Multilinear commutators can also be regarded as Lie algebra commutators, which can define multilinear identities. The following lemma shows that multilinear commutators are compatible with the Mal'cev correspondence.

Lemma 4. *Let G be a radicable locally nilpotent torsion-free group, and L a locally nilpotent Lie algebra over $\mathbb{Q}$ that is in the Mal'cev correspondence with G, with the same underlying set. If $\varkappa$ is a multilinear commutator, then the verbal subgroup $\varkappa(G)$ coincides as a set with the T-ideal $\varkappa(L)$, that is, the ideal of L generated by all values of $\varkappa$ on elements of L (in fact, the subspace spanned by these values).*

Proof. Induction on the weight of $\varkappa$. For weight 1 we have $\varkappa(G) = G = L = \varkappa(L)$ as sets. Now let $\varkappa = [\varkappa_1, \varkappa_2]$. By the induction hypothesis, $\varkappa_i(G) = \varkappa_i(L)$. This also implies that the $\varkappa_i(G)$ are radicable subgroups, since the $\varkappa_i(L)$ are clearly invariant under multiplication by scalars in $\mathbb{Q}$, which is equivalent to taking rational powers in the group. The commutator subgroup $\varkappa(G) = [\varkappa_1(G), \varkappa_2(G)]$ is the smallest normal subgroup of G modulo which $\varkappa_1(G)$ and $\varkappa_2(G)$ commute. As already mentioned in the proof of Lemma 3, this subgroup is also automatically radicable. In the Lie algebra, to this subgroup there corresponds an ideal of L. The quotient group and the quotient Lie algebra are then in the Mal'cev correspondence too. Since elements in the group commute if and only if they commute in the corresponding Lie algebra, this ideal must be the smallest ideal modulo which $\varkappa_1(L)$ and $\varkappa_2(L)$ commute. But this ideal is obviously equal to $[\varkappa_1(L), \varkappa_2(L)] = \varkappa(L)$ (which is in fact the space spanned by all commutators $[a, b]$, $a \in \varkappa_1(L)$, $b \in \varkappa_2(L)$). □

Proof of Theorem 3. Recall that we are given a locally nilpotent torsion-free group G and a normal subgroup H of G satisfying a multilinear commutator identity $\varkappa(H) = 1$ such that G/H has finite rank r (we abbreviate the latter condition to "H has finite co-rank r in G"). We must find a characteristic subgroup C of G that satisfies the same identity $\varkappa(H) = 1$ and has finite $(r, \varkappa)$-bounded co-rank.

Let $\sqrt{G}$ be the Mal'cev completion of G. Let $\sqrt{H}$ be the subgroup of $\sqrt{G}$ obtained by adjoining all roots of the elements of H, so $\sqrt{H}$ is the Mal'cev completion of H. Then $\sqrt{H}$ is a normal subgroup of $\sqrt{G}$. By Lemma 3 the hypothesis $\varkappa(H) = 1$ implies that $\varkappa(\sqrt{H}) = 1$.

The quotient $\sqrt{G}/\sqrt{H}$ is the Mal'cev completion of $(G\sqrt{H})/\sqrt{H}$ and therefore the rank of $\sqrt{G}/\sqrt{H}$ is equal to the rank of $(G\sqrt{H})/\sqrt{H}$; see [1]. The rank of $(G\sqrt{H})/\sqrt{H}$ clearly does not exceed the rank of G/H. Suppose that we find a radicable characteristic subgroup C_1 of $\sqrt{G}$ that satisfies the identity $\varkappa(C_1) = 1$ and has finite $(r, \varkappa)$-bounded co-rank in $\sqrt{G}$. Since every automorphism of G can be extended to an automorphism of $\sqrt{G}$, then $C = G \cap C_1$ will be the required characteristic subgroup of G satisfying the identity $\varkappa(C) = 1$ and having finite $(r, \varkappa)$-bounded co-rank, because the rank of $G/C \cong (GC_1)/C_1$ is at most the rank of $\sqrt{G}/C_1$.

Thus, we can assume from the outset that both $G = \sqrt{G}$ and $H = \sqrt{H}$ are radicable groups, and seek the required subgroup also among radicable ones. Let L be a locally nilpotent Lie algebra over $\mathbb{Q}$ that is in the Mal'cev correspondence with G, and let I be the ideal corresponding to H. By Lemma 4 a normal radicable subgroup S of G satisfies the multilinear commutator identity $\varkappa(S) = 1$ if and only if the corresponding ideal J of L satisfies "the same" identity $\varkappa(I) = 0$. The rank of a radicable locally nilpotent torsion-free group is equal to the length of a normal series with factors isomorphic to $\mathbb{Q}$; see [1]. Therefore the rank of G/S is equal to the codimension of J in L.

Thus, we have the equivalent problem about the Lie $\mathbb{Q}$-algebra L: given an ideal I satisfying the identity $\varkappa(I) = 0$ with quotient L/I of dimension r we need to find an automorphically-invariant ideal J satisfying the same identity $\varkappa(J) = 0$ with quotient L/J of finite (r, c)-bounded dimension. Then the same set J will be the required characteristic subgroup C of G that satisfies the identity $\varkappa(C) = 1$ with the rank of G/C being the codimension of J in L. It remains to apply Theorem 1 to complete the proof of Theorem 3. □

5. Examples

In this section we construct a nilpotent Lie ring of nilpotency class 2 that has an abelian ideal with quotient of rank 2 as an additive group but does not have an automorphically-invariant abelian ideal with quotient of finite rank. (Recall that a group has rank at most r if every finitely generated subgroup can be generated by r elements.) Thus, Theorems 1, 2 cannot be extended to algebras not over fields—even to Lie rings (algebras over integers). We also produce a similar example of a nilpotent group of class 2 that has a normal abelian subgroup with quotient of rank 2 but does not have a characteristic abelian subgroup with quotient of finite rank. This example shows that in [4, Theorem 1.2] the hypothesis of the subgroup being torsion-free or periodic cannot be dropped. (A similar example was independently and almost simultaneously constructed by H. Smith.)

Example 1. For each $i = 1, 2, \ldots$ let $F_i = \langle a_i, b_i \mid [[a_i, b_i], a_i] = 0 = [[a_i, b_i], b_i]\rangle$ be a free class 2 nilpotent Lie algebra over $\mathbb{Q}$ with free generators a_i, b_i. But in what follows we consider the F_i as Lie rings (algebras over $\mathbb{Z}$), so, in particular, F_i is no longer finitely generated. Let $p_1, p_2, \ldots$ be distinct prime numbers. For each i we set

$$K_i = \left\{ \frac{m}{n}[a_i, b_i] \;\middle|\; m, n \in \mathbb{Z}, n \text{ is coprime to } p_i \right\};$$

$$A_i = \left\{ \frac{m}{n}a_i \;\middle|\; m, n \in \mathbb{Z}, n \text{ is coprime to } p_i \right\};$$

$$B_i = \left\{ \frac{m}{n}b_i \;\middle|\; m, n \in \mathbb{Z}, n \text{ is coprime to } p_i \right\}.$$

Consider the Lie ring $L_i = F_i/K_i$. Henceforth, let bars denote the images of elements or subsets in $L_i = F_i/K_i$. The Lie subring $N_i = \overline{A}_i + \overline{B}_i + [L_i, L_i]$ is abelian, since $[x, y] \in K_i$ for any $x \in A_i$ and $y \in B_i$. The presence of the summand $[L_i, L_i]$ ensures that N_i is an ideal of the Lie ring L_i. The additive group of the quotient L_i/N_i is isomorphic to $C_{p_i^\infty} \oplus C_{p_i^\infty}$, where $C_{p_i^\infty}$ is the quasicyclic p_i-group.

Finally we construct the required Lie ring L as the direct sum $L = \bigoplus_i L_i$. Let $N = \bigoplus_i N_i$; this is an abelian ideal of L. The additive group of the quotient L/N has rank 2. We claim that L has no automorphically-invariant abelian ideals with quotient of finite rank.

To simplify notation, for an element $g \in L$ we denote by $\mathbb{Q}g$ the additive subgroup $\{rg \mid r \in \mathbb{Q}\}$. Recall that the additive group of L is the direct sum

$$L = \bigoplus_i \mathbb{Q}\bar{a}_i \oplus \mathbb{Q}\bar{b}_i \oplus \mathbb{Q}\overline{[a_i, b_i]}. \tag{19}$$

For $g \in L$ let $g_{i1}\bar{a}_i$ and $g_{i2}\bar{b}_i$, where $g_{i1}, g_{i2} \in \mathbb{Q}$, be the projections of g onto $\mathbb{Q}\bar{a}_i$ and $\mathbb{Q}\bar{b}_i$ with respect to (19), so that $g = \sum_i g_{i1}\bar{a}_i + g_{i2}\bar{b}_i (\mathrm{mod}[L, L])$.

Suppose that I is an automorphically-invariant ideal of L such that the quotient L/I has finite rank as an additive group. Then there must exist an index i such that the projections of some elements u, v (not necessarily different) onto $\mathbb{Q}\bar{a}_i$ and $\mathbb{Q}\bar{b}_i$ are non-trivial: $u_{i1} \neq 0 \neq v_{i2}$ (actually, there must be many such indices, but we need only one). Then we can produce elements in I—the images of u, v under certain automorphisms of L—that do not commute "in the ith component." Namely, for $s \in \mathbb{Z}$ let φ_s be the automorphism of L_i defined by

$$\varphi_s : \begin{cases} \overline{ra}_i \longrightarrow p_i^s \overline{ra}_i, & r \in \mathbb{Q}, \\ \overline{rb}_i \longrightarrow p_i^{-s} \overline{rb}_i, & r \in \mathbb{Q}. \end{cases}$$

Each of the φ_s can also be regarded as an automorphism of L acting trivially on the L_j for $j \neq i$. We claim that for some $s, t \in \mathbb{Z}$ the ith component of $[\varphi_s(u), \varphi_t(v)]$ is a non-trivial element of $[L_i, L_i]$ and therefore $[\varphi_s(u), \varphi_t(v)] \neq 0$. Indeed, we can, for example, choose negative s so as to make the denominator of $p_i^s u_{i1}$ as a reduced fraction to be divisible by p_i, while simultaneously making the denominator of the reduced fraction $p_i^{-s} u_{i2}$ to be coprime to p_i; and similarly, we choose positive t so as to make the denominator of the reduced fraction $p_i^t v_{i1}$ to be coprime to p_i, simultaneously making the denominator of the reduced fraction $p_i^{-t} v_{i2}$ to be divisible by p_i. Then in the commutator $[\varphi_s(u), \varphi_t(v)]$ the ith component is

$$\big[\big(p_i^s u_{i1}\bar{a}_i + p_i^{-s} u_{i2}\bar{b}_i\big), \big(p_i^t v_{i1}\bar{a}_i + p_i^{-t} v_{i2}\bar{b}_i\big)\big] = p_i^s u_{i1} p_i^{-t} v_{i2}\overline{[a_i, b_i]} + p_i^{-s} u_{i2} p_i^t v_{i1}\overline{[b_i, a_i]}.$$

The first summand on the right is a non-trivial element of $[L_i, L_i]$ by the construction of $L_i = F_i/K_i$, while the second summand is trivial.

Example 2. For each $i = 1, 2, \dots$ let $F_i = \langle a_i, b_i \mid [[a_i, b_i], a_i] = 1 = [[a_i, b_i], b_i]\rangle$ be a free class 2 nilpotent radicable group (with exponents in $\mathbb{Q}$) with free generators a_i, b_i. But in what follows we consider the F_i as abstract groups. Let $p_1, p_2, \dots$ be distinct prime numbers. For each i we set

$$K_i = \big\{[a_i, b_i]^{m/n} \bigm| m, n \in \mathbb{Z},\ n \text{ is coprime to } p_i\big\};$$
$$A_i = \big\{a_i^{m/n} \bigm| m, n \in \mathbb{Z},\ n \text{ is coprime to } p_i\big\};$$
$$B_i = \big\{b_i^{m/n} \bigm| m, n \in \mathbb{Z},\ n \text{ is coprime to } p_i\big\}.$$

Consider the abstract group $G_i = F_i/K_i$. Henceforth, let bars denote the images of elements or subsets in $G_i = F_i/K_i$. The subgroup $N_i = \overline{A}_i\overline{B}_i[G_i, G_i]$ is abelian, since $[x, y] \in K_i$ for any $x \in A_i$ and $y \in B_i$. The presence of $[G_i, G_i]$ ensures that N_i is a normal subgroup of G_i. The quotient G_i/N_i is isomorphic to $C_{p_i^\infty} \times C_{p_i^\infty}$, where $C_{p_i^\infty}$ is the quasicyclic p_i-group.

We construct the required group G as the direct product $G = \prod_i G_i$. Let $N = \prod_i N_i$; this is an abelian normal subgroups of G. The quotient G/N has rank 2. We claim that G has no characteristic abelian subgroups with quotient of finite rank.

For an element $g \in G$ we denote by $g^{\mathbb{Q}}$ the subgroup $\{g^r \mid r \in \mathbb{Q}\}$. Then

$$G = \prod_i \bar{a}_i^{\mathbb{Q}} \times \bar{b}_i^{\mathbb{Q}} \times \overline{[a_i, b_i]}^{\mathbb{Q}}. \tag{20}$$

For $g \in G$ let $\bar{a}_i^{g_{i1}}$ and $\bar{b}_i^{g_{i2}}$, where $g_{i1}, g_{i2} \in \mathbb{Q}$, be the projections of g onto $\bar{a}_i^{\mathbb{Q}}$ and $\bar{b}_i^{\mathbb{Q}}$ with respect to (20), so that $g = \prod_i \bar{a}_i^{g_{i1}} \bar{b}_i^{g_{i2}} \pmod{[G, G]}$.

Suppose that C is a characteristic subgroup of G such that the quotient G/C has finite rank. Then there must exist an index i such that the projections of some elements u, v (not necessarily different) onto $\bar{a}_i^{\mathbb{Q}}$ and $\bar{b}_i^{\mathbb{Q}}$ are non-trivial: $u_{i1} \neq 0 \neq v_{i2}$. Then we can produce elements in C—the images of u, v under certain automorphisms of G—that do not commute "in the ith component." Namely, for $s \in \mathbb{Z}$ let φ_s be the automorphism of G_i defined by

$$\varphi_s : \begin{cases} \overline{a_i^r} \longrightarrow \overline{a_i^r}^{\,p_i^s}, & r \in \mathbb{Q}, \\ \overline{b_i^r} \longrightarrow \overline{b_i^r}^{\,p_i^{-s}}, & r \in \mathbb{Q}. \end{cases}$$

Each of the φ_s can also be regarded as an automorphism of G acting trivially on the G_j for $j \neq i$. We claim that for some $s, t \in \mathbb{Z}$ the ith component of $[\varphi_s(u), \varphi_t(v)]$ is a non-trivial element of $[G_i, G_i]$ and therefore $[\varphi_s(u), \varphi_t(v)] \neq 1$. Indeed, we can, for example, choose negative s so as to make the denominator of $p_i^s u_{i1}$ as a reduced fraction to be divisible by p_i, while simultaneously making the denominator of the reduced fraction $p_i^{-s} u_{i2}$ to be coprime to p_i; and similarly, we choose positive t so as to make the denominator of the reduced fraction $p_i^t v_{i1}$ to be coprime to p_i, simultaneously making the denominator of the reduced fraction $p_i^{-t} v_{i2}$ to be divisible by p_i. Then in the commutator $[\varphi_s(u), \varphi_t(v)]$ the ith component is

$$\left[\bar{a}_i^{p_i^s u_{i1}} \bar{b}_i^{p_i^{-s} u_{i2}}, \bar{a}_i^{p_i^t v_{i1}} \bar{b}_i^{p_i^{-t} v_{i2}}\right] = \overline{[a_i, b_i]}^{\,p_i^s u_{i1} p_i^{-t} v_{i2}} \, \overline{[b_i, a_i]}^{\,p_i^{-s} u_{i2} p_i^t v_{i1}}.$$

The first factor on the right is a non-trivial element of $[G_i, G_i]$ by the construction of $G_i = F_i/K_i$, while the second factor is trivial.

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